Q.1

consider the Weak form.

$$\int_0^L \left[2\frac{du}{dx}\frac{dv}{dx} + 5\frac{du}{dx}v + 10uv \right] dx - \int_0^L v\,dx + 2v(0) = 0$$

Integrating the first term by parts

$\boxed{2}$

$$2\frac{du}{dx}\cdot v\Big|_0^L - \int_0^L 2\frac{d^2u}{dx^2}\cdot v\,dx + \int_0^L \left[5\frac{du}{dx}\cdot v + 10uv\right]dx - \int_0^L v\,dx + 2v(0) = 0.$$

$\boxed{1}$

$$-2\frac{du}{dx}\cdot v(0) + \cancel{2\frac{du}{dx}\cdot v(L)}^{\,0} - \int_0^L v\left[2\frac{d^2u}{dx^2} - 5\frac{du}{dx} - 10u + 1\right]dx + 2v(0) = 0$$

$\boxed{2}$

$$v(0)\left[2 - 2\frac{du}{dx}\right]\Bigg|_{x=0} - \int_0^L v\left[2\frac{d^2u}{dx^2} - 5\frac{du}{dx} - 10u + 1\right]dx = 0$$

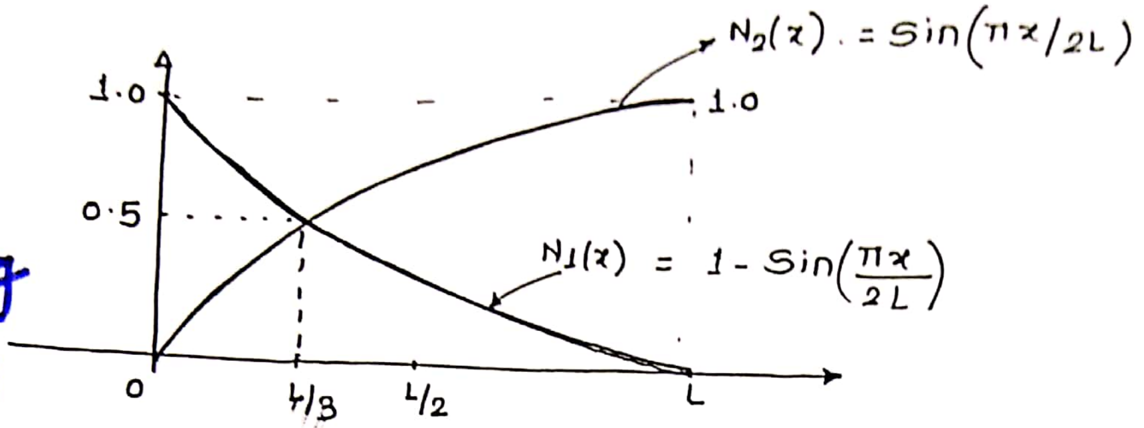
$\boxed{2}$

$$2\frac{d^2u}{dx^2} - 5\frac{du}{dx} - 10u + 1 = 0$$:- Governing differential equation.

$\boxed{2}$ $\frac{du}{dx}\Big|_{x=0} = 1$ $\quad u\Big|_{x=L} = u_L$:- Boundary conditions $\boxed{1}$

Q.2

1)



~~2~~ 1

Properties of the Shape function

1) The delta function property

1

$$N_1\Big|_{x=x_1} = 1 \qquad N_1\Big|_{x=x_2} = 0$$

$$N_2\Big|_{x=x_1} = 0 \qquad N_2\Big|_{x=x_2} = 1.$$

This property is satisfied by the shape functions

2) Partition of unity.

1

$$\sum N_i = 1 \quad \forall x \in [0, L]$$

$$\left[1 - \sin\left(\frac{\pi x}{2L}\right) + \sin\left(\frac{\pi x}{2L}\right)\right] = 1$$

2) $u(x) = [N_1 \; N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [N]\{u\}.$

$u'(x) = [N_1' \; N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [B]\{u\}$

Similarly $w(x) = [w]^T [N]^T$

$w'(x) = [w]^T [B]^T.$

$N_1 = 1 - \sin\frac{\pi x}{2L}$

$N_1' = -\frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right)$

$N_2 = \sin(\pi x/2L)$

$N_2' = \frac{\pi}{2L} \cos(\pi x/2L)$

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Weak form of the governing equation is written as,

$$\int_0^L \frac{dw}{dx} AE \frac{du}{dx}\, dx = \int_0^L q \cdot w\, dx + F_1 w\Big|_{x=0} + F_2 w\Big|_{x=L}$$

By substituting the interpolation scheme & invoking the arbitraryness of $w(x)$.

$$\left[AE \int_0^L [B]^T [B]\, dx \right] \{u\} = \int_0^L [N]^T q\, dx + [F_1 \quad F_2]^T$$

$\boxed{2}$

$$[k]^e = AE \begin{bmatrix} \int_0^L N_1' N_1' dx & \int_0^L N_1' N_2' dx \\ \int_0^L N_2' N_1' dx & \int_0^L N_2' N_2' dx \end{bmatrix}$$

$$= \frac{AE \cdot \pi^2}{8L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

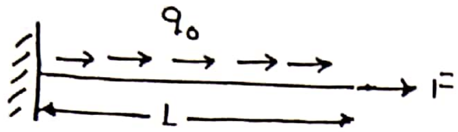
Force vector: $\int_0^L N^T q\, dx = \begin{bmatrix} \frac{qL(\pi-2)}{\pi} \\ \frac{2L \cdot q}{\pi} \end{bmatrix}$

The element level equation is

$$\frac{AE\pi^2}{8L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{qL(\pi-2)}{\pi} \\ \frac{2qL}{\pi} \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$\boxed{2}$

Test case



For assessing the continuity of primary & secondary variables, it is appropriate to consider a mesh of (at least) two elements.

For polynomial interpolation functions the global equation is written as [Assuming the length of the element as $L/2$]

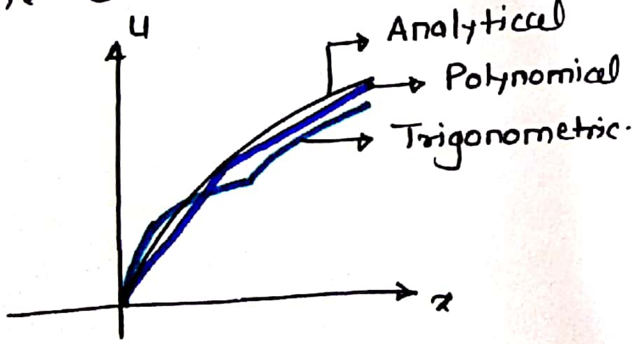
$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} q_0L/4 \\ q_0L/2 \\ q_0L/4 \end{bmatrix} + \begin{bmatrix} R \\ 0 \\ F \end{bmatrix}$$

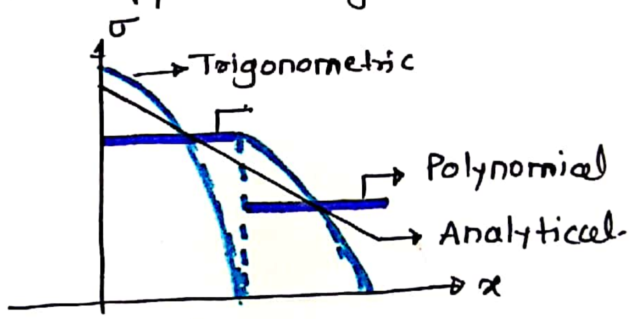
$\boxed{1}$

while for trigonometric interpolation functions,

$$\frac{AE\pi^2}{8(L/2)}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} q_0L(\pi-2)/2\pi \\ q_0L/2 \\ q_0L/\pi \end{bmatrix} + \begin{bmatrix} R \\ 0 \\ F \end{bmatrix}$$

The solutions can be sketched approximately as





$\boxed{1}$

⊕ continuity of primary variable can be attained by using the trigonometric trial functions, However continuity of the derived variable is not achievable.

$\boxed{1}$

⊕ The trigonometric functions lack the requirement of completeness [$\Sigma N_i x_i \neq x$] & hence perform poorly in convergence.

Q.3

1. The body force due to the own weight of the column can be treated as the distributed load with intensity equal to $\rho . A . g$ [Force / length].

considering the upward direction as +ve x the differential equation is written as

$\boxed{1}$ $$AE \frac{d^2u}{dx^2} - \rho A g = 0 \quad \text{with} \quad u\Big|_{x=0} = 0 \quad AEu'\Big|_{x=L} = -P$$

2. The analytical solution is

$$AEu'' = \rho A g$$

$$AEu' = \rho A g . x + C_1$$

$$AEu = \rho A g . \frac{x^2}{2} + C_1 x + C_2$$

$$u\Big|_{x=0} = 0 \Rightarrow C_2 = 0$$

$$AEu'\Big|_{x=L} = \rho A g . L + C_1 = -P$$

$$C_1 = -P - \rho A g L.$$

$\boxed{2}$ $$u = \frac{1}{AE}\left[\rho A g . \frac{x^2}{2} - Px - \rho A g L x\right]$$

$$u_{exact} = \frac{-Px}{AE} + \frac{\rho A g}{2AE}\left[x^2 - 2Lx\right]$$

$$u'_{exact} = \frac{-P}{AE} + \frac{\rho A g}{2AE}\left[2x - 2L\right]$$

$$\sigma_{exact} = \frac{-P}{A} + \frac{\rho A g}{A}\left[x - L\right]$$

$$\sigma\Big|_{x=0} = -892000 \ N/m^2$$

$$\sigma\Big|_{x=L/2} = -846000 \ N/m^2$$

$$\sigma\Big|_{x=L} = -800000 \ N/m^2$$

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The two element finite element equation

$$\frac{AE}{(L/2)}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}q_0L/4\\ q_0L/2\\ q_0L/4\end{bmatrix} + \begin{bmatrix}R\\ 0\\ -P\end{bmatrix}$$

$\boxed{2}$

where $q_0 = -\rho A g$

$$u_1 = 0 \qquad u_2 = \frac{-PL}{2AE} + \frac{3q_0L^2}{8AE} \qquad u_3 = \frac{-PL}{AE} + \frac{q_0L^2}{2AE}.$$

Putting the parameter values, we get.

$\boxed{2}$ $u_1 = 0$ $\qquad u_2 = -8.69\times10^{-5}\ m$ $\qquad u_3 = -1.69\times10^{-4}\ m.$

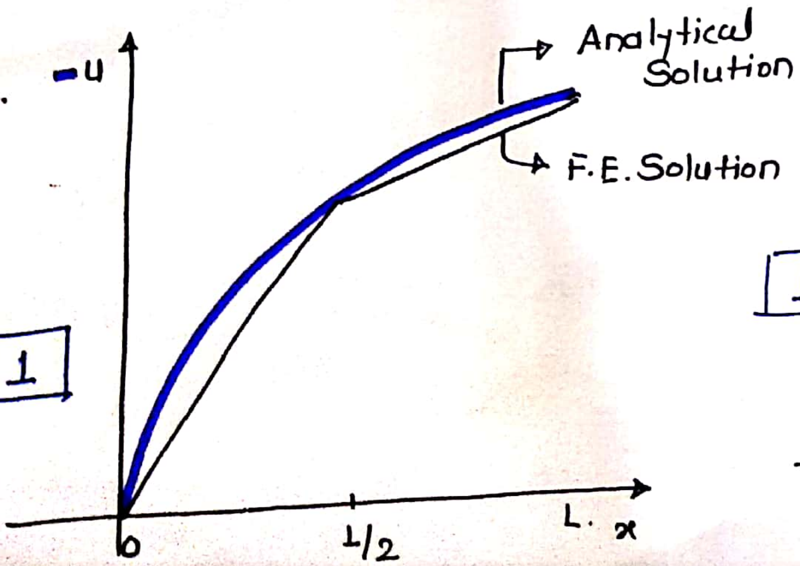
The stresses in the individual element are

$$\sigma_1 = \frac{u_2-u_1}{(L/2)}\cdot E = -869000\ N/m^2$$

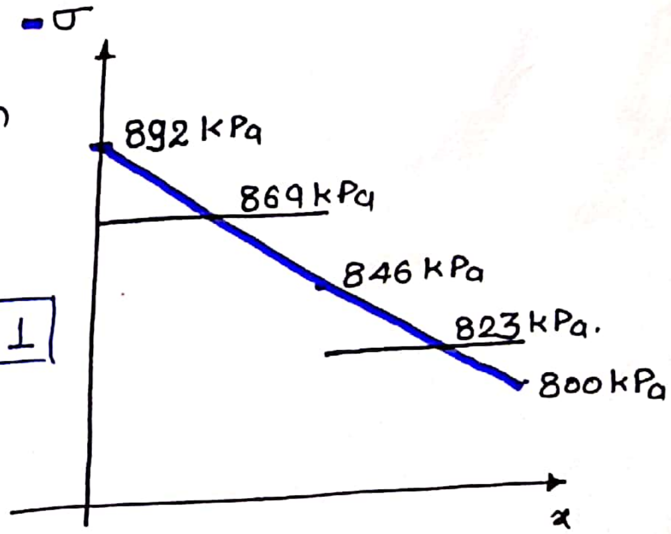
$\boxed{1}$

$$\sigma_2 = \frac{u_3-u_1}{(L/2)}\cdot E = -823000\ N/m^2.$$

4.



$\boxed{1}$



$\boxed{1}$